ON THE EXPRESSIBILITY OF REAL NUMBERS

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ABSTRACT. Mathematics is a field that breathes formality. The very foundations of mathematics are the formal axioms that allow us to confirm, with confidence, the validity of our proofs. But the formality of mathematics has surprising limitations, and in this paper we look at how mathematics is formalized and then explore another dimension of its limitations—namely, the apparent fact that there are real numbers that we cannot express.

Section 2 looks at the foundations of mathematics and builds up the basic knowledge that will be required for Section 3, where the above claim is proven and its implications are explored.

The authors have deliberately made this paper accessible—albeit with some effort on the reader's part—to anyone with an understanding of high school mathematics. The content of Section 2 will no doubt be familiar to experienced readers, although such readers may still find the presentation of the material to be enlightening and entertaining.

1. Three Questions

Does a set that contains all sets that do not contain themselves contain itself? Bertrand Russell's proposition of this famous paradox in Cantor's naïve set theory threw mathematics into a frenzy, and Zermelo's subsequent developments in set theory were one of the first steps into formalizing mathematics.

Nowadays, foundational mathematicians have formally crafted axiomatic systems that are both expressive enough to render valid all the results of modern-day mathematics, yet restrictive enough that no similar foundation-breaking paradoxes have been found to date. It is certainly true that some topics in mathematics, such as the Axiom of Choice or the potential of type theory to replace set theory, are deeply controversial—but they do not carry any implications about the consistency of mathematics. Whether numbers are types, sets, or something completely different affects not what can be done with them but rather the perspectives from which we regard them.

More fundamental to mathematics than the notion of a "number," as is so often the misconception, are the answers to three questions:

1.1. **Question.** What is a valid mathematical object?

1.2. **Question.** What is a valid mathematical statement?

1.3. **Question.** What is a valid mathematical proof?

The next section is devoted to providing satisfactory answers.

2. Foundations of Mathematics

2.1. **Definition** (Sets)**.** A *set* is a collection of objects. A set may be empty, in which case is it called the *empty set*, which is commonly denoted $\emptyset$ or $\{\}$, or it may contain other sets. If a set x contains the set y and z, we may write x or $\{y, z\}$ to denote the set x. Two sets x and y are equal if and only if every member of x is a member of y and vice versa. For instance, $\{x, x\} = \{x\}$. Objects contained in a set are called *elements* of the set.

2.2. *Remark.* Note that the empty set and the set containing the empty set are not equal, a fact that often confuses students. Sets can be thought of as "boxes." With this image in mind, the distinction is clear: an empty box and a box containing an empty box are certainly different. Note, however, that this visualization has its flaws: a box containing three identical red balls must be considered the same as a box containing one copy of the same ball.

Zermelo-Fraenkel Set Theory (or indeed, Russell's type theory, though that is beyond the scope of this article) answers Question 1.1: *every mathematical object is a set.* This definition is sensible considering how much of math revolves around thinking about sets: the set of natural numbers $\mathbb{N}$, the set of even integers $2\mathbb{Z} = \{2x : x \in \mathbb{Z}\}$, the set of real numbers in the closed interval between 0 and 1 $[0, 1] = \{x \in \mathbb{R} : 0 \leq x \leq 1\}$. But there are other ways that it does not seem so sensible. How can a number be a set? How can an ordered pair be a set? How can functions, which are themselves often handled as objects in their own right in rigorous calculus and algebra, be sets?

2.3. **Example** (Common Set-Theoretic Definitions)**.** The following definitions are not central to this article; rather, they are examples to help the reader understand how mathematical objects can be described using the language of set theory.

- The natural number 0 is represented by the empty set, or equivalently $\emptyset$ or $\{\}$.
  Note that some mathematicians do not include 0 in the natural numbers; here we will be inclusive.
- The natural number 1 is represented by the set containing 0, or equivalently $\{0\}$, or $\{\emptyset\}$, or $\{\{\}\}$.
- The natural number 2 is represented by the set containing 0 and 1, or equivalently $\{0, 1\}$ or $\{0, \{0\}\}$, or $\{\emptyset, \{\emptyset\}\}$, or $\{\{\}, \{\{\}\}\}$.
- The natural number $n+1$ is represented by the set containing exactly every natural number between 0 and n.
- The ordered pair of sets (x, y) is represented by the set containing the set containing x and the set containing x and y, or equivalently $\{\{x\}, \{x, y\}\}$.
- The function $f : A \to B$, $x \mapsto f(x)$ is the ordered pair of A and the ordered pair of B and the set of all ordered pairs $(x, f(x))$, or equivalently, $(A, (B, X))$, or $\{\{A\}, \{A, \{\{B\}, \{B, X\}\}\}\}$ where $X = \{(x, f(x)) : x \in A\} = \{\{\{x\}, \{x, f(x)\}\} : x \in A\}$.

These are typical definitions for numbers, pairs, and functions as sets. The definitions for numbers allow us to easily check for order by just checking membership, those for ordered pairs account for order, and functions have every part of them encoded.

From there, the integers can be defined in terms of the naturals, the rationals in terms of the integers, infinite sequences defined in terms of functions on the naturals, and the reals as equivalence classes of rational-valued Cauchy sequences[1].

[1]See A.13–A.15.

2.4. **Definition** (Mathematical Statements)**.** A *mathematical language* consists of a countable *alphabet* and a set of axiomatic *validity rules.* A *mathematical statement* is a finite string of symbols from such an alphabet that satisfies the validity rules.

2.5. **Example** (First-Order Set Theory)**.** The language of first-order set theory has an alphabet containing the following symbols:

- $\forall$, the universal quantifier, read "for all,"
- $\exists$, the existence quantifier, read "there exists,"
- $\in$, the membership operator, read "is in"
- $=$, the equality operator, read "is the same as,"
- $\neg$, the negation operator, read "it is not the case that,"
- $\vee$, the or operator, read "or,"
- $\wedge$, the and operator, read "and,"
- $\implies$, the implication operator, read "implies,"
- $\iff$, the if-and-only-if operator, read "is equivalent to,"
- (and), parentheses,

and a countable number of variable symbols, and a statement follows some common-sense rules of validity.

In the following statement, then, u refers to the empty set, because there is nothing in u:

$$\forall v \,\neg\, (v \in u)$$

Recall from 2.3 that u also refers to 0.

Basically all of modern mathematics—that is, with the exception of the mathematics of mathematical language—can be expressed in first-order set theory.

2.6. *Remark* (Countable Infinity). Recall that in 2.4, a mathematical language was defined to consist of a *countable* alphabet. What does "countable" mean? The natural numbers are often described to have a "countable" cardinality, or size, which would make it seem like "countable" is just jargon for an infinite quantity. But there is a subtlety with infinity that makes this distinction important.

It is generally fallacious to think of *infinity* as a quantity, and generally better to think of it as *more.* The set of natural numbers and the set of even numbers, for example, both have infinite elements: if you give me any number x, I can give you *more* natural numbers or even numbers than x. And while it is intuitively obvious to find a correspondence between the evens and naturals so that every even corresponds to two naturals, making it seem as though there were more evens than naturals, it is not difficult to find another correspondence so that every natural corresponds to two evens: consider $n \mapsto \{4n, 4n+2\}$. Comparing the size of these two sets directly is not a particularly reasonable idea considering the technicalities of what comparing "more"s would entail, since infinity just means "more."

That being said, there are different types of "more." The natural numbers (and even numbers) can be "placed" "on a line" in some order. Indeed, we call this type of infinity *countable infinity* because there is *some* way to "count" it—if I give you some value (not all values, but any value), by counting in that way, you will reach this value at some point. The integers can similarly be enumerated, simply as 0, 1, -1, 2, -2, 3, -3, ...

There are also *uncountable infinities* that do not have such enumerations. We will show the existence of these in Section 3.

2.7. *Remark* (Countable Alphabet). Particularly critical readers may wonder why an alphabet must be countable. It does not necessarily have to be, but it is more meaningful and interesting to define it as such for the following reasons:

- The theorem this paper aims to prove is true and its proof is valid on every sufficiently-expressive language with countable alphabet, and its proof is invalid for any language with uncountable alphabet and the theorem itself in fact trivially false for at least one language with uncountable alphabet;
- The purpose of a *language* is for a mode of expression and communication between two or more people, and it is meaningless to have more distinct, unique symbols in a language than is human-comprehensible, which an uncountably-infinite alphabet certainly would be; and
- First-order set theory can be used to express basically everything in modern mathematics, and it only uses a countable alphabet—indeed, a language which is just as expressive but uses an uncountably-infinite alphabet would only be characterized as redundant.

First-order set theory itself requires a countable alphabet only because it *is* valid to write such statements as $\forall u \, \forall v \, \forall w \, \forall x \, \forall y \, \forall z \, \exists a \, \forall b \, \exists c \, (a = a)$, which although quite meaningless, is still a valid statement. Since statements can be of arbitrary (but not infinite) length, the alphabet must be able to accommodate that.

Having addressed Question 1.2, we proceed to Question 1.3.

2.8. **Definition** (Axiomatic Logic). A *mathematical proof* is a derivation of a formal statement, or *proposition*, through applying "common-sense" rules assumed to be true—*axioms*—to beginning assumptions, called *premises*.

One such axiom allows us to conclude that, for any propositions P, Q, if $P \vee Q$ (at least one of P and Q is true) and $\neg P$ (P is not true), then Q (Q is true). For example, if we know that it is raining or it is snowing, and we know that it is not raining, we can conclude that it is snowing. A similar axiom allows us to conclude that if $\neg P$ (P is not true), then $\neg(P \wedge Q)$ (P and Q are not both true). Another axiom allows us to conclude a logical equivalence of a premise, such as those given by De Morgan's Laws: $\neg(P \wedge Q) \iff (\neg P \vee \neg Q)$ and similarly, $\neg(P \vee Q) \iff (\neg P \wedge \neg Q)$; or the contrapositive: $(P \implies Q) \iff (\neg Q \implies \neg P)$.

2.9. *Remark* (Mathematical Truth). This notion of axiomatic logic implies that a mathematical proof does *not* "prove" that a proposition is "true;" it merely shows that, given one's initial assumptions and a set of derivation rules, one must reach a certain conclusion. Although abstract mathematics provides a useful and interesting tool for describing, exploring, and modelling the real world, absolute truth is beyond its scope: the conclusions at which one arrives through mathematical reasoning are only as strong as one's axioms—the things one considers true by definition.

2.10. **Example** (The Principle of Explosion). By assuming contradictory premises, one can prove any statement.

- Let Q be the statement we wish to prove. Q could be, for instance "Socrates is immortal," "there exists positive integers x, y, z satisfying $x^4 + y^4 = z^4$," or "all members of the empty set are blue."
- Assume $P \wedge \neg P$. (A statement P and its negation $\neg P$ are both true.)
- Then P. (Since $P \wedge \neg P$ is true, both are true and in particular P is true.)

- Then $P \vee Q$. (Since P is true, at least one of P and Q is true.)
- But $\neg P$. (P is not true for the same reason that P is true)
- Therefore, Q. (Since at least one of P and Q is true, and P is not true, Q is true.)

Thus, the desired statement follows from our assumptions.

2.11. *Remark* (Properties of the Empty Set). The third statement in 2.10—"all members of the empty set are blue"—is actually true, which is somewhat counter-intuitive. Suppose it is false. Then there must exist a non-blue member of the empty set. But there are no members of the empty set, and in particular no non-blue member exists. This is logically equivalent to saying that all members of the empty set are blue. Similarly, one can show that all members of the empty set are green, orange, or any colour one likes; indeed, one can show that all members of the empty set are *not* blue. These statements are *vacuously true.*

This provides an answer to Question 1.3. We are now equipped to tackle the problem of inexpressible real numbers.

3. Limits of Formal Systems

With a formal infrastructure for formality, some brilliant and particularly insane mathematicians took to finding its limits. Kurt Gödel published his famous two incompleteness theorems in 1931, and following him were many others who showed various incapabilities of formal mathematical systems. Notable results include

- Gödel's First Incompleteness Theorem: in any sufficiently expressive system (in particular, those whose languages that support "arity of at least two"), there are statements that *are true but cannot be proven so*;
- Gödel's Second Incompleteness Theorem: any consistent, sufficiently expressive system *cannot demonstrate its own consistency*;
- Tarski's Undefinability Theorem: arithmetic truth *cannot be defined* in terms of arithmetic;
- Church's proof that Hilbert's Entscheidungsproblem is *unsolvable*—not that it's true, not that it's false, not that it's badly-defined, not that no one has ever found a solution, but that the problem *cannot* be solved; and
- Turing's proof that the Halting Problem is undecidable.

This paper aims to prove another result about the limits of formal systems: that in a system *both* expressible enough *and* restrictive enough, that there are objects that exist within the system but cannot be expressed. Normally, proofs of existence are accomplished by a direct example; but we are proving that there are numbers that cannot be expressed and thus cannot be directly given as an example. As such, we use indirect means.

This result has been proven before, but we the authors have not found any literature relating to its additional implications.

We aim to prove that there are real numbers that cannot be expressed in first-order set theory, even though first-order set theory is sufficient to express basically all of modern mathematics. In this chapter, we will build up the necessary tech to do just that, and then explore its implications.

Extending this proof to any language that expressive enough to describe all of basic arithmetic, algebra, and calculus (that is, so that the real numbers exist as we commonly know them) but restricted to a countable alphabet is left as an exercise to the reader.

3.1. *Remark.* By framing this in first-order set theory, we are essentially showing a stronger version of the claim that there are sets that exist but are inexpressible.

3.2. **Definition** (Expressible Numbers)**.** Let x be a real number and L a mathematical language. x is *expressible* in L if there is a mathematical statement $F \in L$ that describes x *uniquely*. That is, formally, there is $F \in L$ so that if $F(x)$ (that is, the relation F is satisfied by x), then for all y, $F(y) \implies x = y$ (that is, if another y satisfies F then y is actually x). Conversely, a statement L *expresses* a real number x if it describes it uniquely.

3.3. **Example.** The following examples should help the reader understand the notion of expressibility.

- 1 is expressible by 2.3;
- In general, any natural number is expressible by 2.3;
- Any integer is expressible since (by 2.3) they are each directly defined with respect to exactly one natural number and at most one binary choice;
- Any rational is expressible since (by 2.3) they are each directly defined with respect to exactly two integers;
- $\sqrt{2}$ is expressible because we have notation for it—alternatively, we can express it as "x such that $x \times x = 2$ and $0 < x$" since 2 and 0 are expressible;
- π is expressible as the 6 times the square root of the convergent infinite series $\sum_{k=1}^{\infty} \frac{1}{k^2}$, among other representations; and
- $\int_{-\pi}^{e^2} (x^2 + e^{-x^2})\, dx$ is expressible because π, e, 2, and integrals are expressible; but
- "x such that $x \times x = 2$" does *not* express a number because it describes both the square root of 2 and the negative square root of 2, failing the uniqueness requirement; and in particular,
- "x such that $x \in \mathbb{R}$" does not express a number because it fails the uniqueness requirement.

3.4. **Lemma** (Countable Union of Countable Sets)**.** *The countable union of countable sets is countable. That is, if $s_0, s_1, s_2, ...$ are countable sets, then the set S with $s_n \subseteq S$ for all $n \in \mathbb{N}$ is countable.*

Proof. We find an enumeration for S. We know that $s_0, s_1, s_2, ...$ have enumerations, say $\sigma_0, \sigma_1, \sigma_2, ...$ Then we make an enumeration σ on S so that

$$\sigma = (\sigma_0(0), \sigma_0(1), \sigma_1(0), \sigma_0(2), \sigma_1(1), \sigma_2(0), ...),$$

skipping duplicates.

Intuitively, we make an infinite table of the elements in $s_0, s_1, s_2, ...$ and draw diagonal lines along them, skipping duplicates. Each diagonal lines hits a finite number of elements and these lines can be connected end-to-end, putting every element of S on a line. Therefore S is enumerable and thus countable. □

3.5. **Example** (Countability of the Rationals)**.** The rational numbers can be thought of as the union of every set of integers divided by some nonzero integer; that is, the union of the integers divided by 1, the integers divided by -1, the integers divided by 2, etc.

But the integers are countable (by 2.6) so the rationals are a countable union of countable sets, implying that the rationals are also countable.

3.6. **Corollary** (Statements in First-Order Set Theory)**.** *There are countable mathematical statements in first-order set theory.*

Proof. We consider the number of finite *strings of symbols* from the alphabet of first-order set theory, and we note that every mathematical statement in first-order set theory is also a finite string of symbols from its alphabet, so there must be at most as many mathematical statements as finite strings of symbols in first-order set theory.

We first show inductively that for any length n, the number of finite strings of exactly that length is countable.

Consider the number of finite strings of length 1. This is just the alphabet of first-order set theory, so these are countable.

Then suppose that the number of finite strings of length n is countable. Then each finite string of length $n+1$ is exactly defined from exactly one finite string of length n and one symbol from the alphabet. This is the same situation as in 3.5 and so by 3.4, the finite strings of length $n+1$ is countable.

(Essentially, in this induction we show the length 1's are countable, and then since length 1's are countable, so are length 2's, and length 3's, etc.)

But the number of finite strings of symbols is the union of the sets of the strings of symbols of length n for all n, which is the countable union of countable sets, which by 3.4 is itself countable, and we are done. □

3.7. **Theorem** (Cantor)**.** *The set of real numbers is uncountable.*

Proof. We show a stronger result: that the real numbers between 0 and 1 are uncountable. (You can convince yourself that showing this immediately proves the theorem).

Recall from 2.10 that if one supposes a contradiction, then they can prove anything. We do exactly this, supposing the false and then finding something ridiculous.

We suppose for a contradiction that there is an enumeration on the reals between 0 and 1, say

$$\begin{aligned} x_1 &= 0.a_1a_2a_3a_4a_5a_6... \\ x_2 &= 0.b_1b_2b_3b_4b_5b_6... \\ x_3 &= 0.c_1c_2c_3c_4c_5c_6... \\ x_4 &= 0.d_1d_2d_3d_4d_5d_6... \\ &\text{etc.} \end{aligned}$$

Then we make a new real number x between 0 and 1 from this enumeration so that

$$x = 0.y_1y_2y_3y_4y_5y_6...$$

so that if $a_1 = 5$ then $y_1 = 6$ and $y_1 = 5$ otherwise, and if $b_2 = 5$ then $y_2 = 6$ and $y_2 = 5$ otherwise, etc.

But $x \neq x_1$ since the first digit doesn't match, and $x \neq x_2$ since the second digit doesn't match, and so on, so it cannot possibly appear in the enumeration. But since all numbers between 0 and 1 must appear in the enumeration, x cannot possibly be between 0 and 1; that is, $x < 0$ or $1 < x$. But $0 < x < 1$ by its very construction. Then by transitivity, $x < 0 < x$ or $x < 1 < x$, and in particular, $x < x$, which is clearly absurd.

But we did not make any unwarranted steps in our proof except for our supposition that there was an enumeration, so it must be the case that there is no such enumeration, and so the reals between 0 and 1 are uncountable. □

3.8. *Remark* (A Notion of Order on Infinities). From the proof of 3.7, notice that there were *more* reals between 0 and 1 than are countable, so the uncountable infinity is *more* than the "more" of the countable infinity.

3.9. **Proposition** (Inexpressibility). *There are real numbers that cannot be expressed in first-order set theory, even though first-order set theory is sufficient to express the rest of modern mathematics. Formally, there exists $x \in \mathbb{R}$ so that there does* not *exists a statement F in first-order set theory that expresses it.*

Proof. By 3.6, there are countable mathematical statements in first-order set theory. In particular there are fewer statements that express a number than there are statements in general, so there are at most countable mathematical statements that express a number in first-order set theory.

Since each such expressing statement expresses exactly one number, there are then at most a countable number of real numbers that are expressible.

But by 3.7 there are uncountable real numbers, and by 3.8, uncountable quantities are greater than countable quantities.

Then the set of inexpressible real numbers cannot possibly be empty. Thus, there exist real numbers that cannot be expressed in first-order set theory. □

3.10. **Corollary** (Inexpressible Superiority). *There are more inexpressible reals than expressible reals.*

Proof. We show that the inexpressibles are uncountable, and the proof follows immediately from 3.8.

Suppose for a contradiction that the inexpressibles are countable. We know that the reals are the union of the inexpressibles and the expressibles. By 3.9 we know that the expressibles are countable.

Then the reals would be countable since we can take one element at a time each from the expressibles and inexpressibles, just as the integers in 2.6 did from the naturals and the negatives of the naturals. But by 3.7 we know that the reals are uncountable, which is a contradiction. □

3.11. **Definition** (Expressible Functions). Let $f : A \to B$ be a function. f is *expressible* in first-order set theory if there is a statement F in first-order set theory that uniquely describes the set from 2.4 corresponding to f, and we say that such F *expresses* f.

3.12. **Lemma** (Inexpressible Functions). *There are inexpressible functions.*

Proof. Consider the set of all constant functions $\mathbb{R} \to \mathbb{R}$. Since there are inexpressible numbers, there are functions in this set that sends its parameter to an inexpressible number. But since that number cannot be expressed, neither can the set of ordered pairs with it as its second element, and so neither can the set corresponding to a function that works on that set of ordered pairs, though it exists in the set of all real-valued functions over the reals. □

3.13. **Lemma** (Expressive Idempotency). *Let f be an expressible function whose domain is E. If $x \in E$ is expressible, then so is $f(x)$.*

Proof. Since f and x are both expressible, $f(x)$ can be expressed by the statement $y = f(x)$. □

The following is a somewhat surprising restatement of 3.13; though it is the direct contrapositive, it does not appear nearly as obviously and gives rise to interesting possibilities.

3.14. **Corollary** (Inexpressibles from Inexpressibles). *Let f be an expressible function. If $f(x)$ is inexpressible, then x is inexpressible.*

Proof. Contrapositive of 3.13. □

3.15. **Corollary** (Bijective Preservation). *Let f be a bijective expressible function. x is expressible if and only if $f(x)$ is expressible.*

Proof. The forwards implication is immediate from 3.13. As for the backwards implication, note that since f is bijective, it admits an inverse f^{-1}, which is expressed by flipping the entries in the set of ordered pairs.

Suppose $f(x)$ is expressible. Then by 3.13, $f^{-1}(f(x)) = x$ is expressible. But $f(x)$ was arbitrary, so we're done. □

3.16. *Remark* (Inexpressible Generation). The above theorem then allows us to *freely generate* inexpressible numbers from any given inexpressible number simply by finding expressible bijective functions that we can put it through.

3.17. **Theorem** (Inexpressible Density). *The set of expressible real numbers is dense in the real numbers.*

Proof. We present a standard ϵ–N proof. The tech, notation, and terminology in this proof are standard definitions in calculus and are explained in Appendix A.

Let β be an arbitrary real number and x be an inexpressible real number chosen from 3.9. We will construct an inexpressible-real-valued sequence from x that converges to β.

We know from the density of the rational numbers that there is a sequence $(a_k)_{k=1}^{\infty} \in \mathbb{Q}^{\mathbb{N}}$ so that $\lim_{k\to\infty} a_k = \beta$. Choose such a sequence.

Using x we construct a sequence of inexpressibles $(b_k)_{k=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$ so that $\lim_{k\to\infty} b_k = 0$. We do this by letting $b_k = x2^{-k}$, and since multiplication by an expressible number is an expressible function and multiplication by a nonzero number is a bijective function, by 3.15 every b_k is inexpressible.

Finally, let $(c_k)_{k=1}^{\infty} \in \mathbb{R}^{\mathbb{N}}$ so that $c_k = a_k + b_k$. Note that since addition by an expressible number is an expressible function and addition by any number is a bijective function, by 3.15 every c_k is inexpressible.

Then let $\epsilon > 0$. Since $\lim_{k\to\infty} a_k = \beta$, there is M_a so that any $m > M_a$ satisfies $|\beta - a_m| < \frac{\epsilon}{2}$. Since $\lim_{k\to\infty} b_k = 0$, there is M_b so that any $m > M_b$ satisfies $|b_m| < \frac{\epsilon}{2}$.

Let $N = \max(M_a, M_b)$. Note that if $n > N$, then we must also have $n > M_a$ and $n > M_b$. Then for all $n > N$, we have

$$\begin{aligned} |\beta - c_n| &= |\beta - a_n + a_n - c_n| && \text{adding zero} \\ &\leq |\beta - a_n| + |a_n - c_n| && \text{by the triangle inequality} \\ &= |\beta - a_n| + |b_n| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

Thus $\lim_{k\to\infty} c_k = \beta$. But $(c_k)_{k=0}^{\infty}$ was a sequence of inexpressible numbers, and β was an arbitrary real number, so every real number is a limit of some sequence of inexpressible numbers, and so the set of inexpressible numbers is dense in the real numbers. □

4. Applications of Expressibility Theory

Recall that an inexpressible number can never be expressed by a mathematical statement. That means that it is impossible to work with these numbers directly. There is no application of inexpressible numbers—they never show up unless we look for them because no other line of derivation allows us to arrive at them.

There are no applications of this work.

And that is beautiful, because that makes this the purest math there is.

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Appendix A. Calculus

This appendix explains the terminology and develops the technology used in Theorem 2.17.

A.1. **Definition** (Absolute Value)**.** Let $x \in \mathbb{R}$. The *absolute value* of x is

$$|x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$$

A.2. *Remark.* Let $x, y \in \mathbb{R}$. You can think of $|x - y|$ as the *distance* between x and y. In particular, for $y = 0$, $|x - y| = |x - 0| = |x|$ is the distance between x and 0.

An equivalent definition of absolute value is $|x| := \sqrt{x^2}$, which you can verify yourself as an exercise.

A.3. **Theorem** (Triangle Inequality)**.** *Let* $a, b, c \in \mathbb{R}$. *Then* $|a - c| \leq |a - b| + |b - c|$.

Proof. Without loss of generality, say $a \leq b$ and $a \leq c$ so $b - a \geq 0$ and $c - a \geq 0$. There are two cases to consider:

Case 1 ($b \leq c$ so $c - b \geq 0$).

$$\begin{aligned} |a-c| &= c - a \\ &= c + 0 - a \\ &= c - b + b - a \\ &= |c-b| + |b-a| \\ &= |a-b| + |b-c| \\ \implies |a-c| &\leq |a-b| + |b-c|. \end{aligned}$$

Case 2 ($b > c$ so $b - c > 0$).

$$\begin{aligned} |a-c| &= c - a \\ &= c - b + b - a \\ &> b - c + b - a \\ &= |b-c| + |b-a| \\ &= |a-b| + |b-c| \\ \implies |a-c| &\leq |a-b| + |b-c| \end{aligned}$$

□

A.4. *Remark.* It may be more intuitive to call the triangle inequality the *detour theorem*: the direct path from A to B is *never longer* than a path from A to C to B.

A.5. *Notation* (Functions). Let A, B be sets. The set of all functions $f : A \to B$ is denoted B^A.

A.6. **Definition** (Sequences)**.** Let A be a set. A *sequence* $(S_n)_{n=0}^{\infty}$ in S, or S-valued sequence, is a function $S : \mathbb{N} \to A$, for which we use the following special notation and terminology:

- The n-th *term* of S is $S(n)$, commonly denoted S_n, and we write
- $(S_n)_{n=0}^{\infty} = (S_0, S_1, S_2, ...)$.

A.7. **Definition** (Convergence)**.** Let $(S_n)_{n=0}^{\infty} \in \mathbb{R}^{\mathbb{N}}$ be a real-valued sequence. S *converges* if there is some $\beta \in \mathbb{R}$ so that for all $0 < \epsilon \in \mathbb{R}$ there is $N \in \mathbb{N}$ so that for all $N < n \in \mathbb{N}$, $|\beta - S_n| < \epsilon$.

If a real-valued sequence $(S_n)_{n=0}^{\infty} \in \mathbb{R}^{\mathbb{N}}$ converges to $\beta \in \mathbb{R}$, then we write $\lim_{n\to\infty} S_n = \beta$ and we say the *limit* of $(S_n)_{n=0}^{\infty}$ as n goes to infinity is β.

A.8. **Example.** Consider the following sequences:

- $(A_n)_{n=0}^{\infty} = (0)_{n=0}^{\infty} = (0, 0, 0, 0, ...)$,
- $(B_n)_{n=0}^{\infty} = \left(\frac{1}{n+1}\right)_{n=0}^{\infty} = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...)$,
- $(C_n)_{n=0}^{\infty} = ((-2)^{-n})_{n=0}^{\infty} = (1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, ...)$, and
- $(D_n)_{n=0}^{\infty} = \left(\frac{\sin n}{\ln(n+2)}\right)_{n=0}^{\infty} \doteq (0, 0.766, 0.656, 0.088, -0.422, -0.493, -0.134, 0.299, ...)$.

These sequences all "approach" 0 as n becomes very large. Indeed, these sequences all converge to 0.

Consider the following sequences:

- $(E_n)_{n=0}^{\infty} = (n)_{n=0}^{\infty} = (0, 1, 2, 3, ...)$,
- $(F_n)_{n=0}^{\infty} = (1)_{n=0}^{\infty} = (1, 1, 1, 1, ...)$,
- $(G_n)_{n=0}^{\infty} = ((-1)^n)_{n=0}^{\infty} = (1, -1, 1, -1, ...)$,
- $(H_n)_{n=0}^{\infty} = (\sin n)_{n=0}^{\infty} \doteq (0, 0.841, 0.909, 0.141, ...)$, and
- $(I_n)_{n=0}^{\infty} = (\max\{n(-1)^n, 0\})_{n=0}^{\infty} = (0, 1, 0, 3, 0, 5, ...)$.

Here, only sequence F converges at all, but not to 0. E never stays around anywhere, G never gets close to 0 though it bounces around, and H and I get close to 0 but does not stay close.

From this, we can come up with the following criterion for convergence: a sequence $(S_n)_{n=0}^{\infty}$ converges to β if it can get *as close as you want* to β and then will *stay* that close to it forever. Readers can convince themselves that this is exactly what Definition A.7 accomplishes.

Readers can also verify that $N_A \geq 0$, $N_B \geq \frac{1}{\epsilon}-1$, $N_C \geq -\log_2 \epsilon$, and $N_D \geq e^n-2$ satisfy Definition A.7. This method of finding an N-value for any $\epsilon > 0$ is commonly called an ϵ–N proof.

A.9. **Definition** (Density)**.** Let $A \subseteq B$ be sets. A is *dense* on B if for every $\beta \in B$ there is an A-valued sequence $(S_n)_{n=0}^{\infty} \in A^{\mathbb{N}}$ that converges to β.

A.10. **Theorem.** *Let $S \subseteq \mathbb{R}$. The following are equivalent:*

(1) *S is dense on $\mathbb{R}$.*
(2) *For every $x < y \in \mathbb{R}$ there is $s \in S$ so that $x < s < y$.*

Proof. (1) $\implies$ (2). Let $x < y \in \mathbb{R}$. There is a sequence $(R_n)_{n=0}^{\infty} \in S^{\mathbb{N}}$ that converges to $\frac{x+y}{2}$ and an N so that

$$n > N \implies \left|\frac{x+y}{2} - R_n\right| < \frac{y-x}{2}$$

so choose $r = R_n$ for some $n > N$. Then $x = \frac{x+y}{2} - \frac{y-x}{2} < r < \frac{x+y}{2} + \frac{y-x}{2} = y$.

(2) $\implies$ (1). Let $\beta \in \mathbb{R}$. Let $(a_n)_{n=0}^{\infty} = (\beta)_{n=0}^{\infty} \in \mathbb{R}^{\mathbb{N}}$ and $(b_n)_{n=0}^{\infty} = (\beta + 2^{-n})_{n=0}^{\infty} \in \mathbb{R}^{\mathbb{N}}$. Then by hypothesis for every n there is $c_n \in S$ with $a_n < c_n < b_n$.

Let $\epsilon > 0$, choose $N > -\log_2 \epsilon$, and let $n > N$. Then

$$\begin{aligned} |\beta - c_n| &= |a_n - c_n| \\ &< |a_n - b_n| \\ &= \left|2^{-n}\right| \\ &= 2^{-n} \\ &< 2^{-N} \\ &< 2^{\log_2 \epsilon} \\ &= \epsilon \end{aligned}$$

□

A.11. *Remark.* Let S be a set dense in $\mathbb{R}$. Then S is *everywhere.*

A.12. **Example.** The following sets are dense in $\mathbb{R}$:

- the rationals $\mathbb{Q}$,
- the irrationals $\bar{\mathbb{Q}}$, and
- the dyadics $\left\{\frac{x}{2^y} : x \in \mathbb{Z}, y \in \mathbb{N}\right\}$

A.13. **Definition** (Cauchy Sequences)**.** Let $(S_n)_{n=0}^{\infty} \in \mathbb{R}^{\mathbb{N}}$ be a real-valued sequence. S is a *Cauchy sequence* if for all $0 < \epsilon \in \mathbb{R}$ there is $N \in \mathbb{N}$ so that for all $N < m, n \in \mathbb{N}$, $|S_n - S_m| < \epsilon$.

A.14. **Theorem.** *Let $(S_n)_{n=0}^{\infty} \in \mathbb{R}^{\mathbb{N}}$ be a real-valued sequence, and suppose that $(S_n)_{n=0}^{\infty}$ converges to $L \in \mathbb{R}$. Then S is a Cauchy sequence.*

Proof. Let $\epsilon > 0$ be an arbitrary positive real number. By the definition of convergence, we can choose some $N \in \mathbb{N}$ so that if $n > N$, then $|S_n - L| < \frac{\epsilon}{2}$. Thus, if $m, n > N$, then

$$|S_m - S_n| \leq |S_m - L| + |S_n - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

where the first inequality follows from A.3. But this means precisely that $(S_n)_{n=0}^{\infty}$ is a Cauchy sequence. □

A.15. *Remark.* A sequence is Cauchy if the terms get arbitrarily close together and stay close together forever. In A.14, we showed that every convergent sequence is Cauchy. In fact, the reverse implication is also true: every Cauchy sequence is convergent. The proof of this fact is beyond the scope of this paper.

Cauchy sequences are useful because they give us a criterion for determining whether or not a sequence converges without referring to its limit. This allows us to define real numbers as limits of Cauchy sequences of rational numbers without our definitions being circular.